

Lecture 4: Frequency Moment Estimation in Streams

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Frequency Moments

Assume we have a stream A , of length N which is composed of m different types of items a_1, \dots, a_m each of which repeats itself n_1, \dots, n_m times (in arbitrary order). We define the frequency moments f_k as:

$$f_k = \sum_{i=1}^m n_i^k$$

Our aim is to process the stream one element at a time and attain an (ϵ, δ) -approximation for f_k . That is, a multiplicative factor $(1 \pm \epsilon)$ with probability at least $1 - \delta$. Note that f_0 is the number of distinct elements in the stream m and that f_1 is the number of elements N . f_2 is also an important quantity which represents how “skewed” the distribution of the elements in stream is.

Estimating f_0

Here we describe an algorithm for estimating f_0 which merges (and hopefully simplifies) ideas from [1] and [2]. First, assume a hash function $h : a \rightarrow [0, 1]$ uniformly. Let us define a random variable $X = \min_i h(a_i)$. Intuitively, X should be roughly $1/m$ and therefore $1/X$ should be a fair estimate of m . This is almost true. In what comes next we make this into an exact statement.

Let us first compute the expectation of X . The distribution function f_X of the random variable X is $f_X(x) = m(1-x)^{m-1}$. This is because, we have m different choices for the minimal element and for every value it takes, x , all the rest $m-1$ values need to be higher than it (w.p $(1-x)^{m-1}$). Therefore:

$$\begin{aligned} E[X] &= \int_0^1 xm(1-x)^{m-1}dx \\ &= \int_0^1 (1-y)my^{m-1}dy \\ &= \int_0^1 my^{m-1}dy - \int_0^1 my^m dy \\ &= 1 - \frac{m}{m+1} = \frac{1}{m+1} \end{aligned}$$

This is after the substitution $y = 1 - x$. We now compute the variance of X . For that we first compute

$E[X^2]$.

$$\begin{aligned}
E[X^2] &= \int_0^1 x^2 m(1-x)^{m-1} dx \\
&= \int_0^1 (1-y)^2 m y^{m-1} dy \\
&= \int_0^1 m y^{m-1} dy - \int_0^1 2m y^m dy + \int_0^1 m y^{m+1} dy \\
&= 1 - \frac{2m}{m+1} + \frac{m}{m+2} \leq \frac{2}{(m+1)^2}
\end{aligned}$$

Thus, the standard deviation of $\sigma(X)$ is in the same order of magnitude as its expectation $E[X]$. To reduce this ratio we again define $Y = \frac{1}{s} \sum_{i=1}^s X_i$ for which $E[Y] = \frac{1}{m+1}$. and $Var[Y] \leq \frac{2}{s(m+1)^2}$.

Using Chebyshev's inequality we get that

$$\Pr\left[|Y - \frac{1}{m+1}| \geq \frac{\varepsilon/2}{m+1}\right] \leq \frac{8}{\varepsilon^2 s} \leq \delta$$

if $s \geq \frac{8}{\varepsilon^2 \delta}$. Therefore, multiplying this procedure $\frac{8}{\varepsilon^2 \delta}$ different hash function and taking their mean minimal value guaranties that with probability at least $1 - \delta$ we have $\frac{1}{m+1}(1 - \varepsilon/2) \leq Y \leq \frac{1}{m+1}(1 + \varepsilon/2)$. In other words: $(m+1)\frac{1}{1+\varepsilon/2} \leq \frac{1}{Y} \leq (m+1)\frac{1}{1-\varepsilon/2}$. But, since $\frac{1}{1-\varepsilon/2} \leq 1 + \varepsilon$ and $1 - \varepsilon \leq \frac{1}{1+\varepsilon/2}$ we get the desired results that $(m+1)(1 - \varepsilon) \leq \frac{1}{Y} \leq (m+1)(1 + \varepsilon)$

Estimating f_1

This is basically counting the N elements in the stream. A trivial solution therefore requires $O(\log(n))$ bits of memory. It is also possible to store an approximate counter in the space $O(\log \log(n))$ (see [3]) but we will not discuss this here.

Estimating all Frequency Moments $k > 0$

We follow the derivation in [1]. For now, assume we know N in advance. This is not necessary and we will fix it later. Let us first define a random variable X . We choose an index $q \in [1, \dots, N]$ uniformly at random. Let a be the element in place q in the stream, i.e. $a = A_q$. Define by r the number of times a appears in the stream after location q , including. In other words $r = |\{i | A_i = a, i \geq q\}|$. We define X :

$$X = N(r^k - (r-1)^k)$$

We claim that $E[X] = f_k$. Let us define the variable $e_{i,j}$ which indicates the event that the index q is such that $A_q = a_i$ and a_i appears exactly j times after the location q . Note that the events $e_{i,j}$ are disjoint and that if $e_{i,j}$ happens than r takes the value j . Therefore, $X = \sum_{i,j} e_{i,j} N(j^k - (j-1)^k)$. Moreover, $\Pr[e_{i,j}] = \frac{n_i}{N} \frac{1}{n_i} = \frac{1}{N}$ since the probability of choosing a_i is $\frac{n_i}{N}$ and given that this happens the probability of

each index (out of the locations of a_i) is equal to $\frac{1}{n_i}$.

$$\begin{aligned}
E[X] &= \sum_{i,j} E[e_{i,j} N(j^k - (j-1)^k)] \\
&= \sum_{i=1}^m \sum_{j=1}^{n_i} \Pr[e_{i,j}] N(j^k - (j-1)^k) \\
&= \sum_{i=1}^m \sum_{j=1}^{n_i} (j^k - (j-1)^k) \\
&= \sum_{i=1}^m n_i^k = f_k .
\end{aligned}$$

It is somewhat complicated and tedious to compute the variance of X . Citing from [1] we have that:

$$\text{Var}[X] \leq km^{1-1/k} f_k^2 .$$

We define Y as the mean of s different copies of X , $Y = \frac{1}{s} \sum_{i=1}^s X_i$. Clearly, $E[Y] = E[X] = f_k$ and $\text{Var}[Y] \leq \text{Var}[X]/s = km^{1-1/k} f_k^2/s$. Using Chebyshev's inequality we have that

$$\Pr[|Y - f_k| > \varepsilon f_k] \leq \frac{\text{Var}[Y]}{\varepsilon^2 f_k^2} \leq \frac{km^{1-1/k}}{\varepsilon^2 s}$$

Demanding that $s \geq \frac{km^{1-1/k}}{\varepsilon^2 \delta}$ gives that $\Pr[|Y - f_k| > \varepsilon f_k] \leq \delta$ which concludes the construction.

Estimating f_2

We will give here a better estimator of f_2 . Assume a hash function $h : a \rightarrow \{-1, 1\}$ with probability $1/2$ each. Define $Z = \sum_{i=1}^N h(A_i) = \sum_{i=1}^m n_i h(a_i)$. Consider the variable $X = Z^2$. As usual, we will begin with computing the expectation and variance of X .

$$\begin{aligned}
E[X] &= E[Z^2] = E\left[\sum_{i=1}^m n_i h(a_i)^2\right] \\
&= E\left[\left(\sum_{i=1}^m n_i h(a_i)\right)\left(\sum_{i'=1}^m n_{i'} h(a_{i'})\right)\right] \\
&= \sum_{i=1}^m \sum_{i'=1}^m n_i n_{i'} E[h(a_i)h(a_{i'})] \\
&= \sum_{i=1}^m n_i^2 = f_2
\end{aligned}$$

Similarly,

$$\begin{aligned}
E[X^2] &= E[Z^4] = \sum_{i=1}^m n_i^4 + 6 \sum_{1 \leq i < i' \leq m} n_i^2 n_{i'}^2 \\
\text{Var}[X] &= E[X^2] - E^2[X] \leq 4 \sum_{1 \leq i < i' \leq m} n_i^2 n_{i'}^2 \leq 2f_2
\end{aligned}$$

Finally, defining $Y = \frac{1}{s} \sum_{i=1}^s X_i$, where each X_i is an independent copy of X we have that:

$$\Pr[|Y - f_2| \geq \varepsilon f_2] \leq \delta$$

if $s \geq \frac{2}{\varepsilon^2 \delta}$.

Connection to random projections (next class)

Consider the s hash functions $h_i : a \rightarrow \{-1, 1\}$ we used in estimating the second frequency moment. Consider the matrix $H \in \mathbb{R}^{s \times m}$ such that $H(i, j) = h_i(j)$. Also, consider representing each input element a_i by \vec{a}_i , the i 'th standard basis vector in \mathbb{R}^m (the vector whose i 'th entry is equal to 1 and the rest are zero). Analogously, \vec{A}_i is the vector representing the i 'th element in the stream. Remember that our estimate Y of f_2 was $\frac{1}{s} \sum_{i=1}^s Z_i^2 = \|\frac{1}{\sqrt{s}} \vec{Z}\|^2$. Moreover, from the definition of \vec{Z} , H , and \vec{A}_i we have that $\vec{Z} = \sum_{i=1}^N H \vec{A}_i = H \sum_{i=1}^N \vec{A}_i = H \vec{A}$. Here, $\vec{A} = \sum_{i=1}^N \vec{A}_i = [n_1, n_2, \dots, n_m]$. Note however, that $f_2 = \|\vec{A}\|^2$ by definition of the second frequency moment. We get that for any stream and any element frequencies $\|\frac{1}{\sqrt{s}} H \vec{A}\|^2 \approx_{(\epsilon, \delta)} \|\vec{A}\|^2$. In other words, multiplying the vector \vec{A} by the matrix $\frac{1}{\sqrt{s}} H$ is very likely to preserve its ℓ_2 norm. We will see that this phenomenon is in fact more overarching and has some serious consequences on point ensembles in high dimensional euclidian spaces.

References

- [1] Noga Alon, Yossi Matias, and Mario Szegedy. The space complexity of approximating the frequency moments. In *Proceedings of the twenty-eighth annual ACM symposium on Theory of computing*, STOC '96, pages 20–29, New York, NY, USA, 1996. ACM.
- [2] Edith Cohen. Size-estimation framework with applications to transitive closure and reachability. *J. Comput. Syst. Sci.*, 55:441–453, December 1997.
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